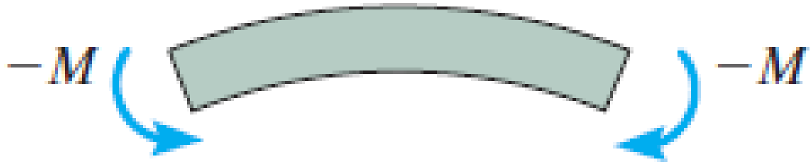
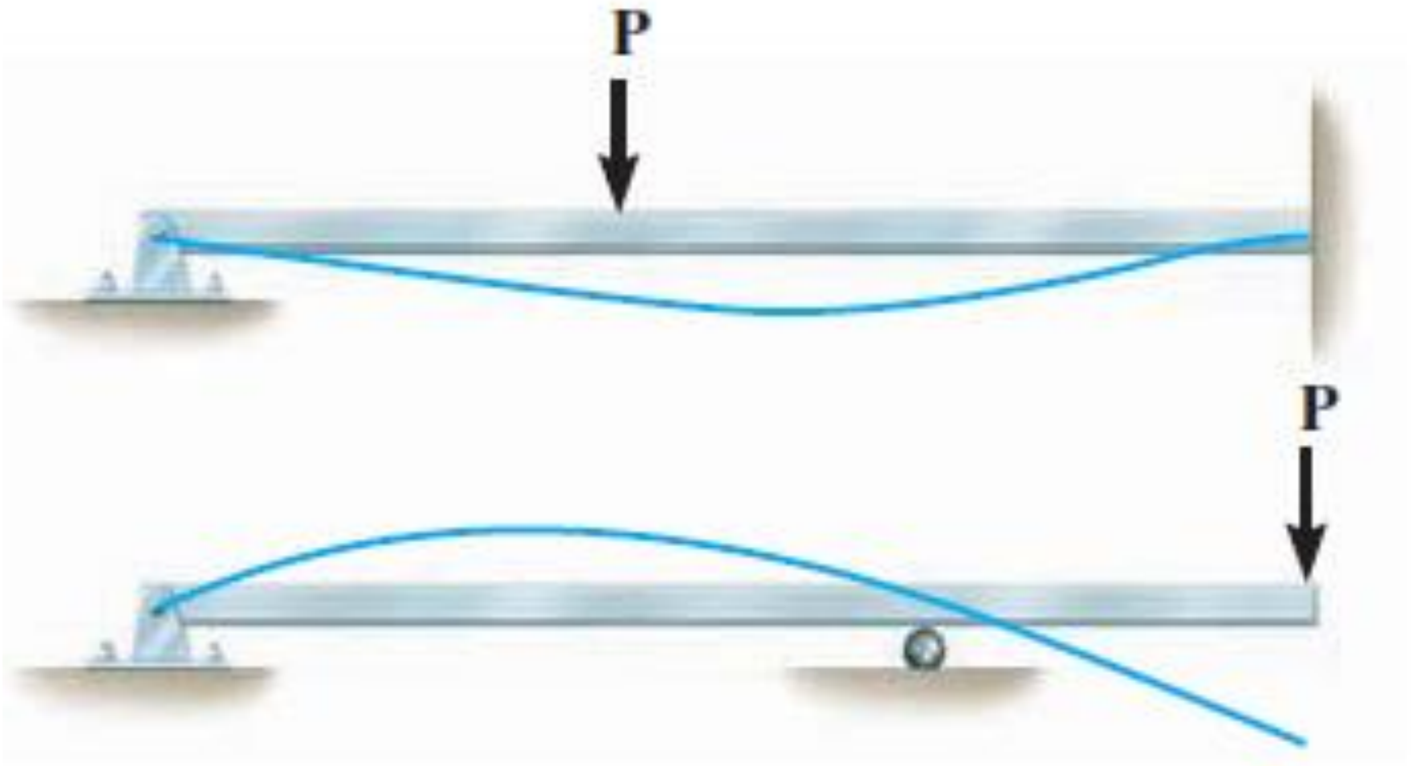
The Elastic Curve

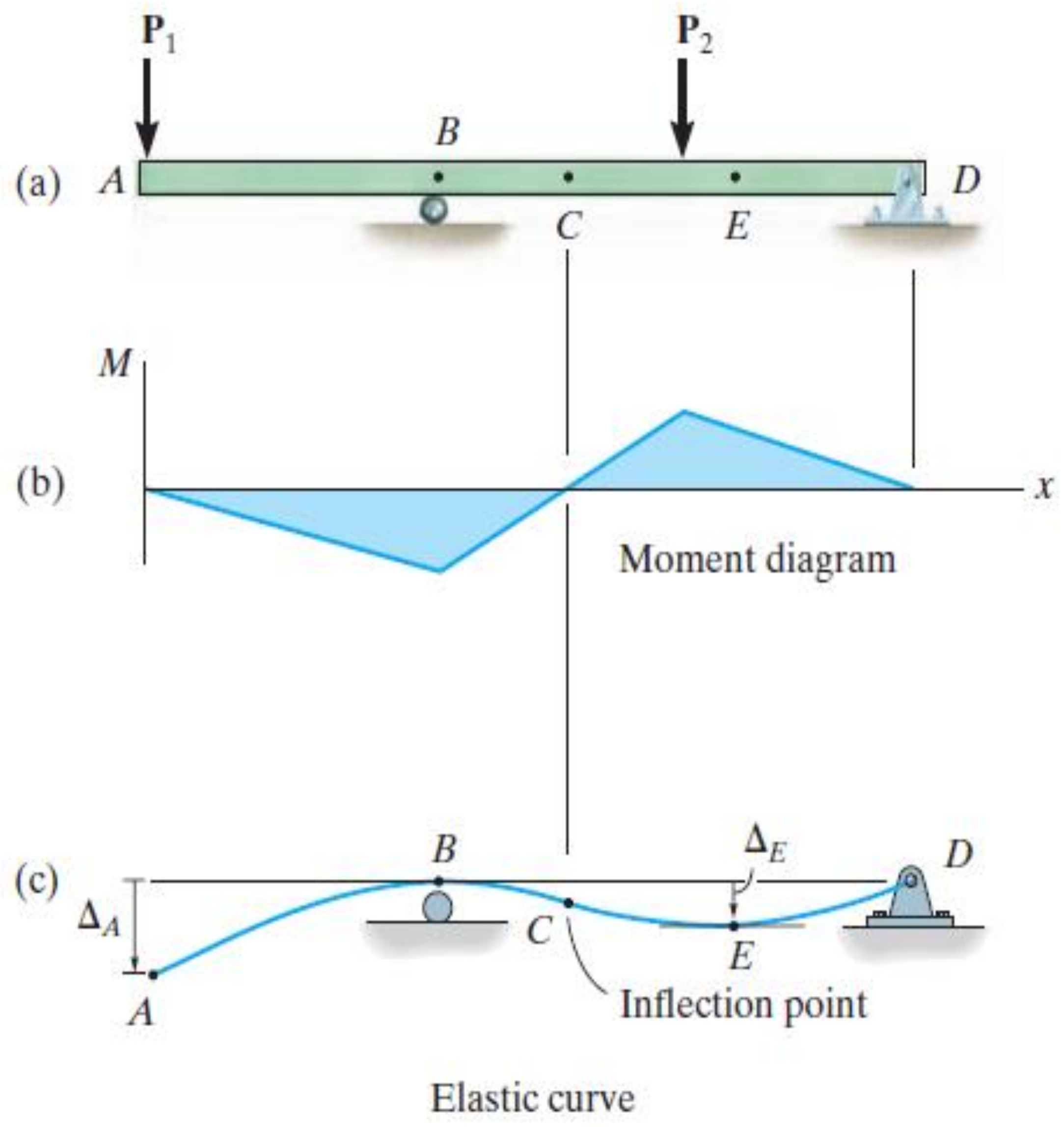
The deflection of a beam or shaft must often be limited in order to provide integrity and stability of a structure or machine, and prevent the cracking of any attached brittle materials such as concrete or glass. Furthermore, code restrictions often require these members not vibrate or deflect severely in order to safely support their intended loading. Most important, though, deflections at specific points on a beam or shaft must be determined if one is to analyze those that are statically indeterminate.

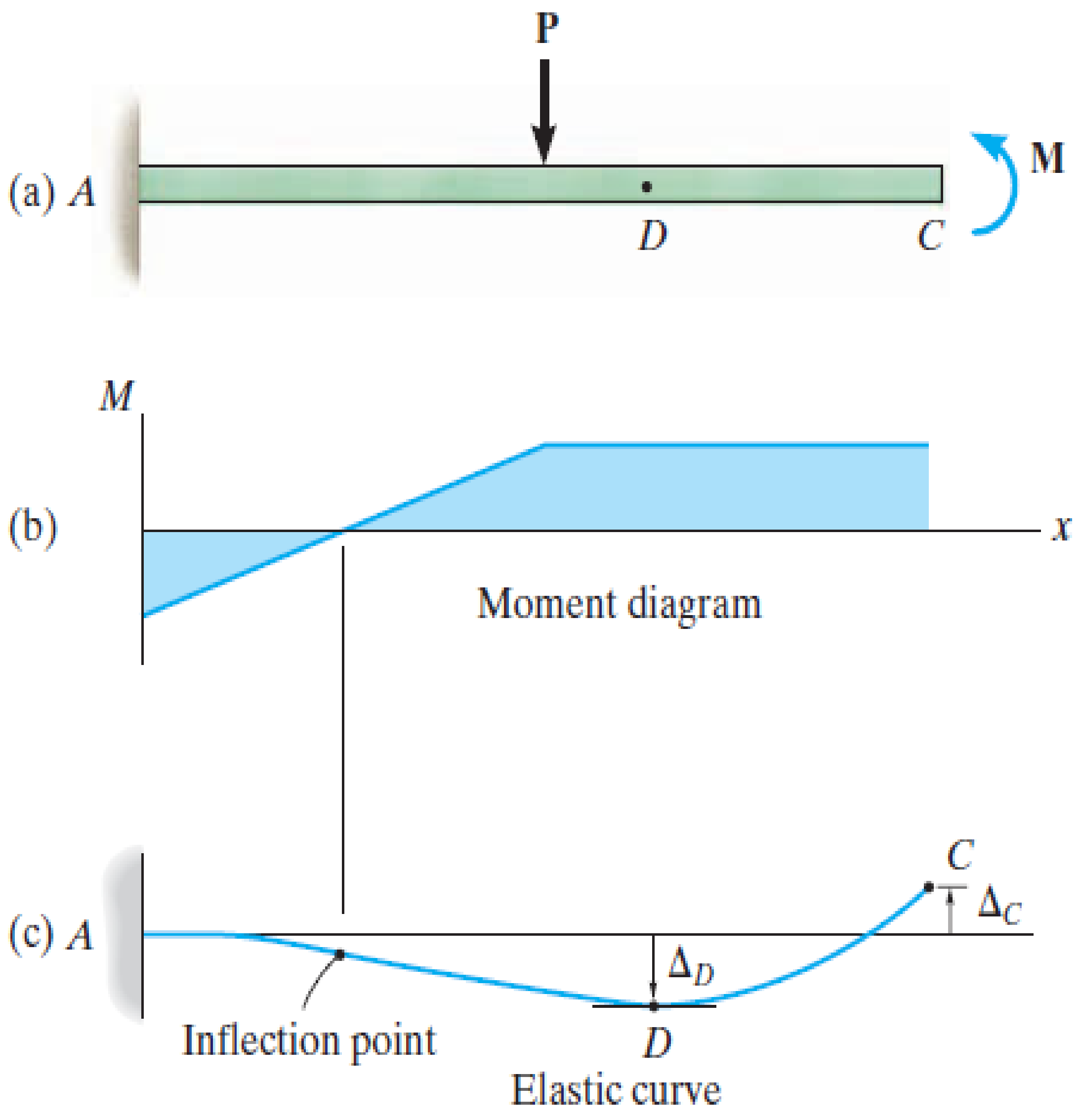
Positive internal moment
concave upwards
(a)



Negative internal moment
concave downwards
(b)



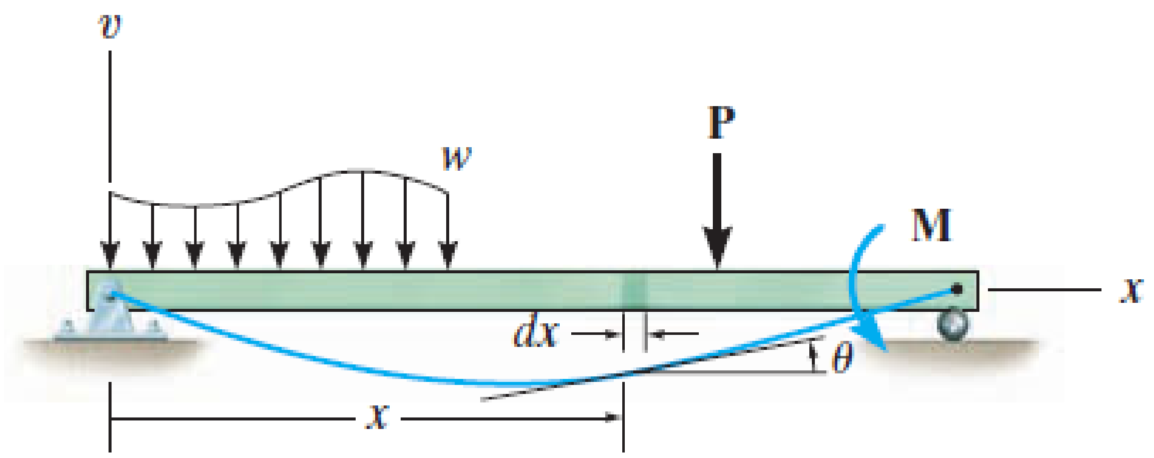

P1
P2
(a) A
B
C
E
D
M
(b)
x
Moment diagram
(c)
ΔA
A
B
C
ΔE
E
D
Inflection point
Elastic curve


P
M
(a) A
D
C
M
(b)
x
Moment diagram
(c) A
C
ΔC
ΔD
D
Inflection point
Elastic curve

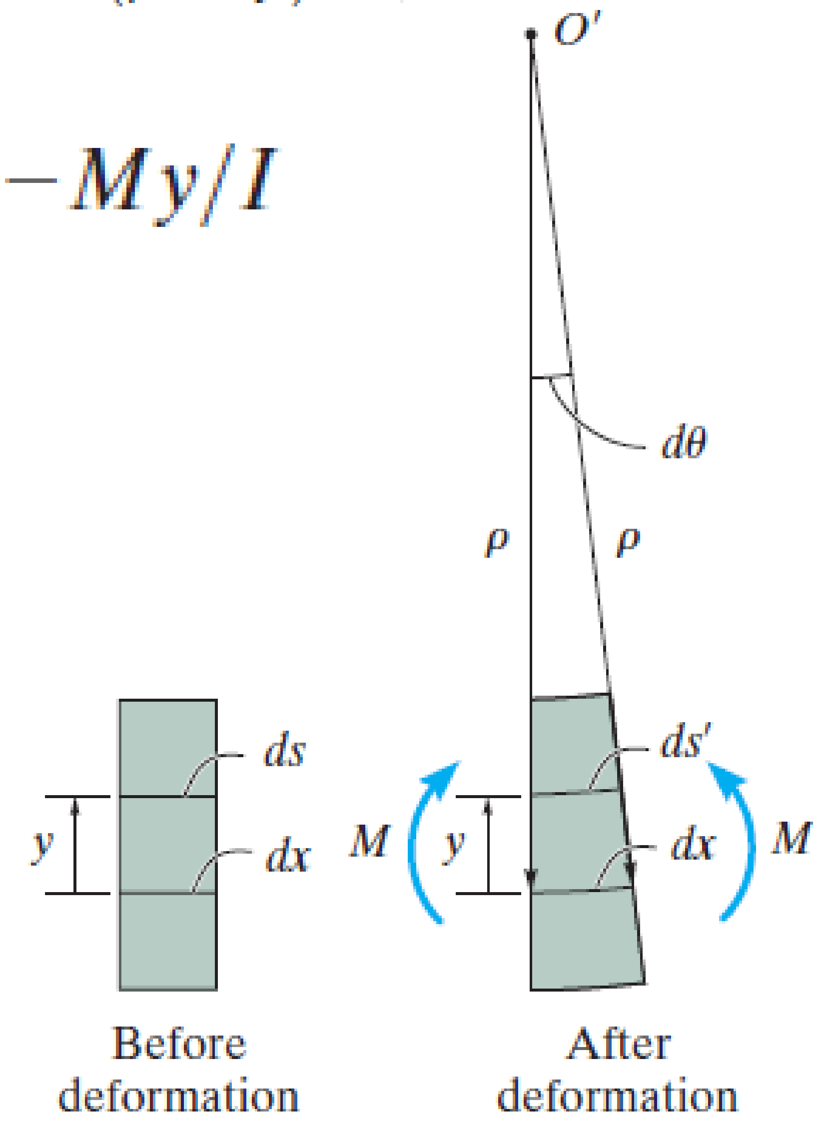
example, the strain in arc ds, located at a position y from the neutral axis, is $\epsilon = (ds' - ds)/ds$. However, $ds = dx = \rho\, d\theta$ and $ds' = (\rho - y)\, d\theta$, and so $\epsilon = [(\rho - y)\, d\theta - \rho\, d\theta]/\rho\, d\theta$ or

$$\frac{1}{\rho} = -\frac{\epsilon}{y} \qquad \epsilon = \sigma/E \qquad \sigma = -My/I$$

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$



(a)



(b)

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

$$\frac{1}{\rho} = -\frac{\sigma}{Ey}$$

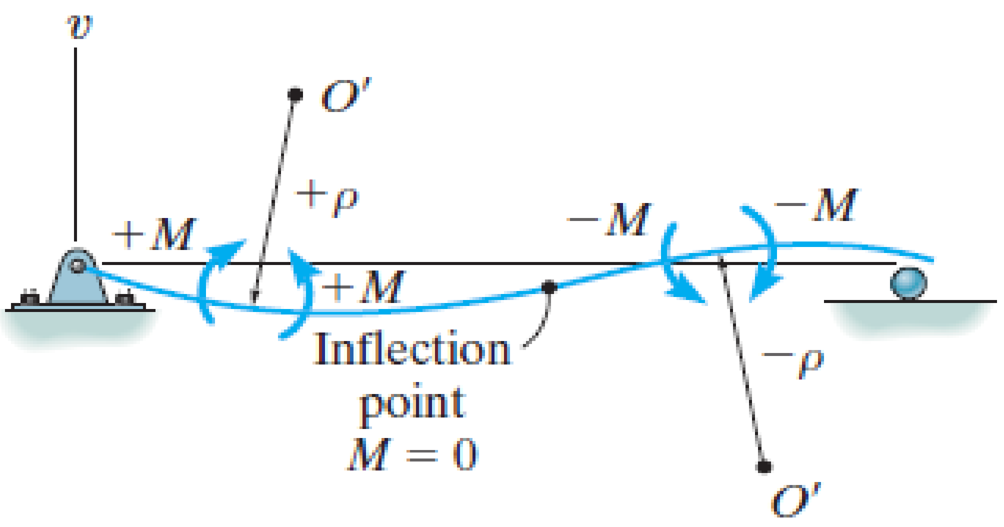
where

ρ = the radius of curvature at the point on the elastic curve ($1/\rho$ is referred to as the *curvature*)

M = the internal moment in the beam at the point

E = the material's modulus of elasticity

I = the beam's moment of inertia about the neutral axis



Slope and Displacement by Integration

The equation of the elastic curve for a beam can be expressed mathematically as $v = f(x)$. To obtain this equation, we must first represent the curvature $(1/\rho)$ in terms of v and x. In most calculus books it is shown that this relationship is

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

$$\frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}} = \frac{M}{EI}$$

The curvature, as defined above, can be *approximated* by

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

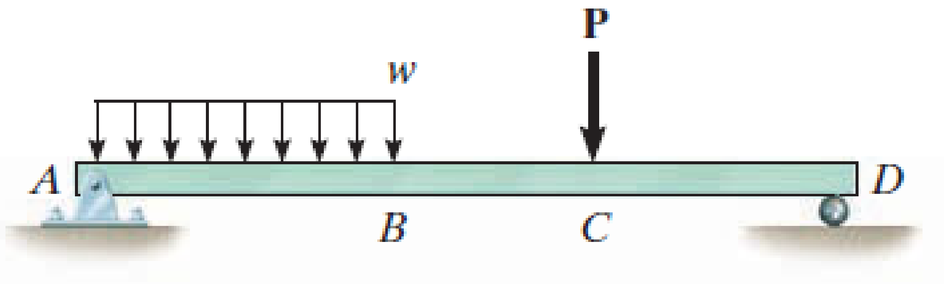
differentiate each side with respect to x and substitute $V = dM/dx$

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x)$$

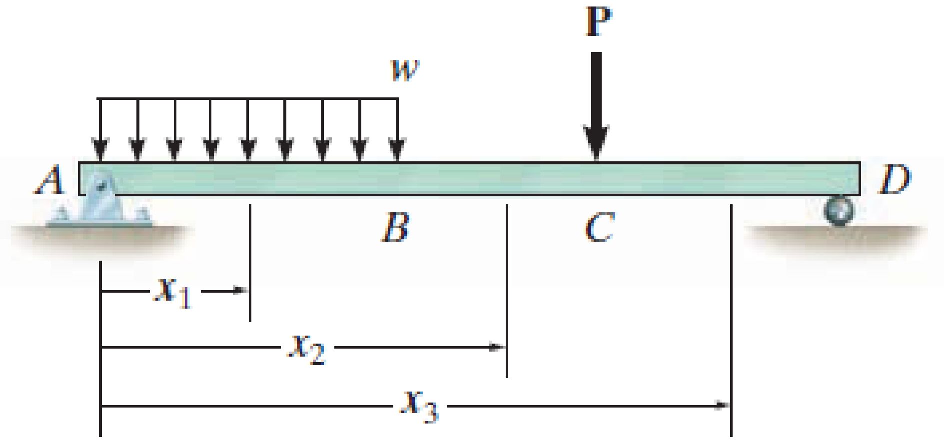
Differentiating again, using $w = dV/dx$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x)$$

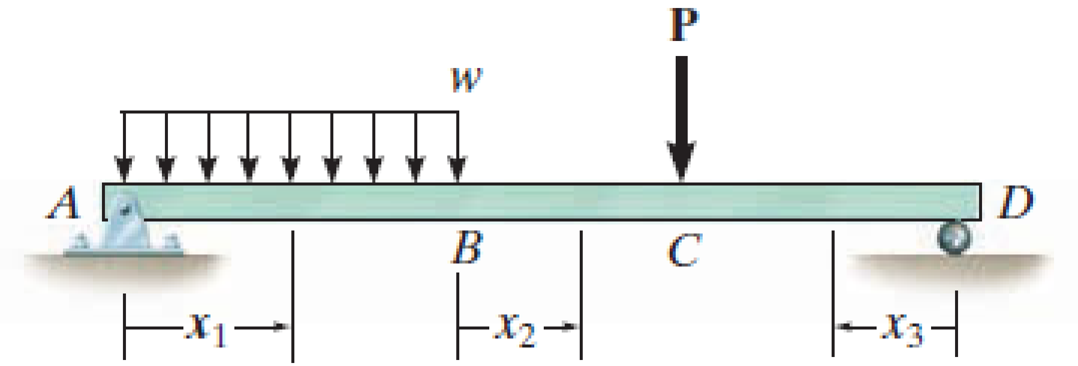
$$\boxed{\begin{aligned} EI\frac{d^4v}{dx^4} &= w(x) \\ EI\frac{d^3v}{dx^3} &= V(x) \\ EI\frac{d^2v}{dx^2} &= M(x) \end{aligned}}$$



(a)



(b)



(c)

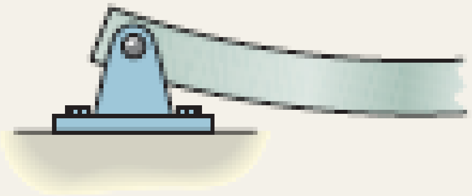
1

$\Delta = 0$

$M = 0$

Roller

2

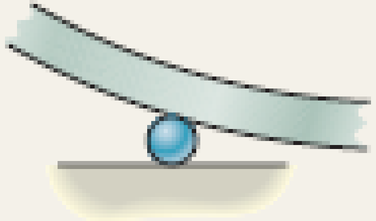


$\Delta = 0$

$M = 0$

Pin

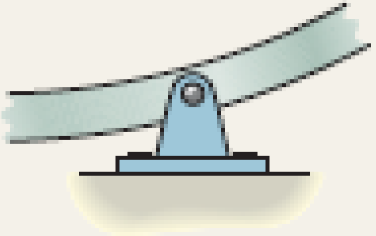
3



$\Delta = 0$

Roller

4



$\Delta = 0$

Pin

5

$\theta = 0$

$\Delta = 0$

Fixed end

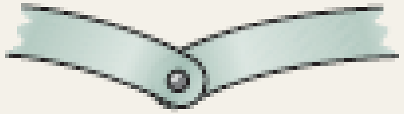
6

$V = 0$

$M = 0$

Free end

7

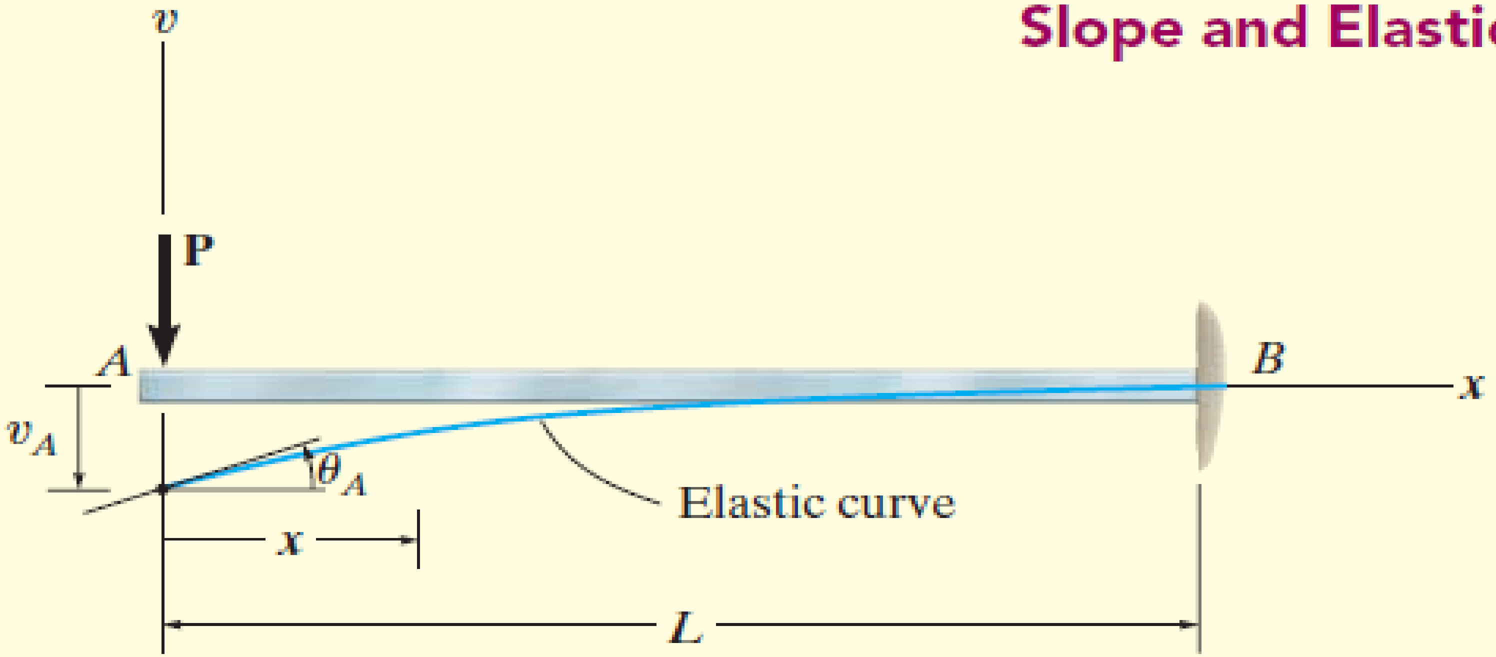


$M = 0$

Internal pin or hinge

Example 1:- The cantilevered beam shown in Figure below is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. *EI* is constant.

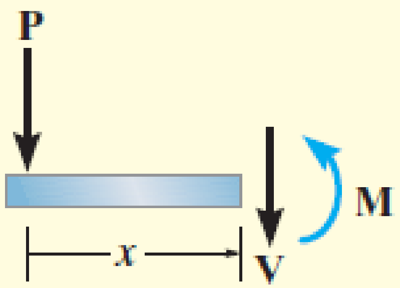
Slope and Elastic Curve.



$$EI\frac{d^2v}{dx^2} = -Px \quad (1)$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \quad (2)$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \quad (3)$$



Using the boundary conditions $dv/dx = 0$ at $x = L$ and $v = 0$ at $x = L$, Eqs. 2 and 3 become

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2$$

Thus, $C_1 = PL^2/2$ and $C_2 = -PL^3/3$. Substituting these results into Eqs. 2 and 3 with $\theta = dv/dx$, we get

$$\theta = \frac{P}{2EI}(L^2 - x^2)$$

$$v = \frac{P}{6EI}(-x^3 + 3L^2 x - 2L^3) \qquad \textit{Ans.}$$

Maximum slope and displacement occur at $A(x = 0)$, for which

$$\theta_A = \frac{PL^2}{2EI} \qquad (4)$$

$$v_A = -\frac{PL^3}{3EI} \qquad (5)$$

SOLUTION II

$$EI\frac{d^4v}{dx^4} = 0$$

$$EI\frac{d^3v}{dx^3} = C_1' = V$$

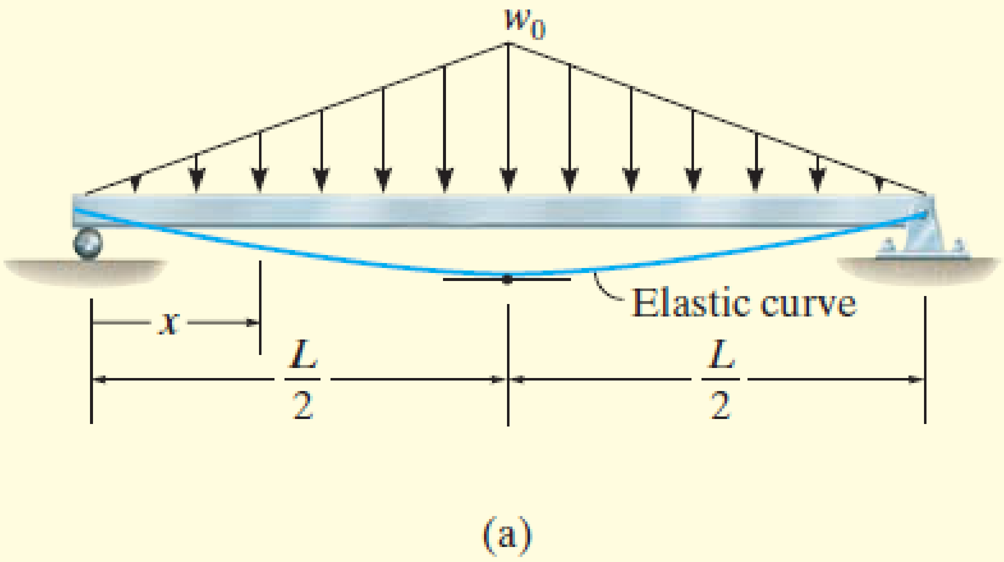
The shear constant can be evaluated at since (negative according to the beam sign convention,). Thus, Integrating again, Here at so and as a result obtains and the solution proceeds as before.

$$EI\frac{d^3v}{dx^3} = -P$$

$$EI\frac{d^2v}{dx^2} = -Px + C_2' = M$$

Here $M = 0$ at $x = 0$, so $C_2' = 0$, and as a result one obtains Eq. 1 and the solution proceeds as before.

The simply supported beam shown in Fig. 12–11*a* supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.



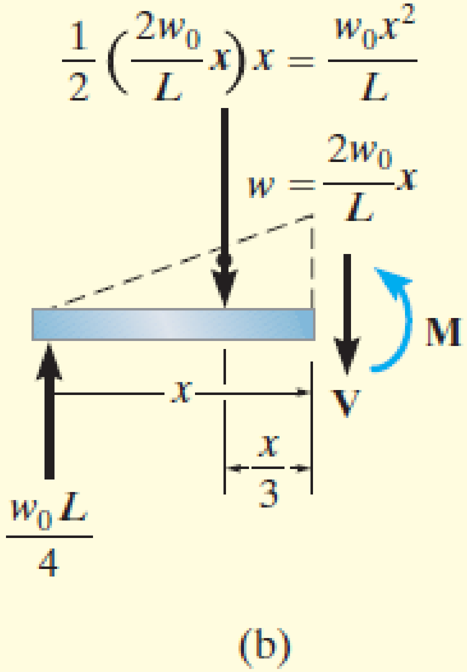


Fig. 12–11

SOLUTION I

Elastic Curve. Due to symmetry, only one x coordinate is needed for the solution, in this case $0 \leq x \leq L/2$. The beam deflects as shown in Fig. 12–11a. The maximum deflection occurs at the center since the slope is zero at this point.

Moment Function. A free-body diagram of the segment on the left is shown in Fig. 12–11b. The equation for the distributed loading is

$$w = \frac{2w_0}{L}x \qquad (1)$$

Hence,

$$\circlearrowright + \Sigma M_{NA} = 0; \qquad M + \frac{w_0 x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0 L}{4}(x) = 0$$

$$M = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4}x$$

Slope and Elastic Curve. Using Eq. 12–10 and integrating twice, we have

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x \qquad (2)$$

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 + C_1$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 + C_1x + C_2$$

The constants of integration are obtained by applying the boundary condition $v = 0$ at $x = 0$ and the symmetry condition that $dv/dx = 0$ at $x = L/2$. This leads to

$$C_1 = -\frac{5w_0L^3}{192} \qquad C_2 = 0$$

Hence,

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 - \frac{5w_0L^3}{192}$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 - \frac{5w_0L^3}{192}x$$

Determining the maximum deflection at $x = L/2$, we have

$$v_{\max} = -\frac{w_0 L^4}{120EI} \qquad \textit{Ans.}$$

SOLUTION II

Since the distributed loading acts downward, it is negative according to our sign convention. Using Eq. 1 and applying Eq. 12–8, we have

$$EI\frac{d^4v}{dx^4} = -\frac{2w_0}{L}x$$

$$EI\frac{d^3v}{dx^3} = V = -\frac{w_0}{L}x^2 + C_1'$$

Since $V = +w_0L/4$ at $x = 0$, then $C_1' = w_0L/4$. Integrating again yields

$$EI\frac{d^3v}{dx^3} = V = -\frac{w_0}{L}x^2 + \frac{w_0L}{4}$$

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x + C_2'$$

Here $M = 0$ at $x = 0$, so $C_2' = 0$. This yields Eq. 2. The solution now proceeds as before.